On the low-dimensional modelling of Stratonovich stochastic differential equations

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Abstract

We develop further ideas on how to construct low-dimensional models of stochastic dynamical systems. The aim is to derive a consistent and accurate model from the originally high-dimensional system. This is done with the support of centre manifold theory and techniques. Aspects of several previous approaches are combined and extended: adiabatic elimination has previously been used, but centre manifold techniques simplify the noise by removing memory effects, and with less algebraic effort than normal forms; analysis of associated Fokker-Plank equations replace nonlinearly generated noise processes by their long-term equivalent white noise. The ideas are developed by examining a simple dynamical system which serves as a prototype of more interesting physical situations.

Keywords: stochastic differential equation, centre manifold, low-dimensional modelling, noisy dynamical system, Fokker-Planck equation.

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1 Introduction

Centre manifold theory is increasingly recognised as providing a rational route to the low-dimensional modelling of high-dimensional dynamical systems. Applications of the techniques have ranged over, for example, triple convection [1], feedback control [3], economic theory [5], shear dispersion [11], nonlinear oscillations [22], beam theory [18], flow reactors [2], and the dynamics of thin fluid films [19]. New insights given by the centre manifold picture enable one to not only derive the dynamical models, but also to provide accurate initial conditions [15, 8], boundary conditions [17], and, particularly relevant to this paper, the treatment of forcing [7].

The above developments have all taken place in the context of deterministic dynamical systems. However, in practice, we want to apply the same concepts in a noisy environment. Just one example of interest is the along-stream dispersion of a contaminant in a turbulent river—the turbulence can only reasonably be modelled by utilising stochastic factors. Thus it is vital that we address the problem of constructing low-dimensional models of stochastic differential equations (SDEs) using centre manifold ideas and techniques.

Boxler [4] has put the subject on a firm base by proving the existence of centre manifolds for SDEs. Further, these are proven relevant as models for the full system; for example the reduced system of evolution equations do indeed predict the stability of a degenerate fixed point. However, the analysis is very sophisticated and the results, as seen in the given examples, involve infinite sums which seem impractical in most applications. In Sections 2 and 3 we show how to straightforwardly construct simpler versions of such centre manifolds.

Earlier work by Schöner & Haken [20, 21] used the concepts of slaving and adiabatic elimination to develop one approach to constructing low-dimensional models of SDEs. Their analysis initially appears very similar, in Section 2 we derive exactly equivalent results, but there are useful improvements through the centre manifold viewpoint. One immediate benefit, in direct contrast to more traditional methods such as that of multiple scales, is that one can be much more flexible about the ordering of various effects in the model. Another is that we present the analysis much more cleanly as an algorithm rather than a hierarchy of formulae; one that can be easily adapted [7, §5] to the vast majority of problems where it is inconvenient to change basis in order to separate the "master modes" from the "slave modes."

Further, the work of Cox & Roberts [7, 15] showed how to use flexibility inherent in the centre manifold approach to remove memory effects of time-dependent forcing from the evolution on the centre manifold (the forcing is then purely local in time), unlike the approach of Schöner & Haken [20, 21]. However, only the leading order effect of the forcing was considered. In a separate line, and following the work of Coullet *et al* [6], Sri Namachchivaya & Lin [13] used the techniques of normal form coordinate transformations, to show how to extract a low-dimensional stochastic centre manifold. The normal form transformation has the same flexibility recognised by Cox & Roberts and consequently also demonstrates the simplification in the evolution that may be attained. Thus, in Section 3 we naturally extend the approach of Cox & Roberts [7] to significantly simplify the models of Schöner & Haken [20]. In large, physically interesting problems, dimensional reduction is achieved via the approach described herein with much less algebra than is needed by the normal form approach.

Lastly, preceding authors have presented their results in terms of noise processes that have the same short-time scale of the transients that we set out to eliminate by the modelling. This is inconsistent. Instead we propose that the irreducible noise processes in the model should be further simplified by only considering them on the long-time scale of the centre manifold evolution. This is obtained, Section 4, from a centre manifold analysis of the Fokker-Planck equation for a noisy dynamical system. However, as also pointed out by Sri Namachchivaya & Lin [13], Knobloch & Wiesenfeld's [10] centre manifold analysis of Fokker-Planck equations needs to be made more systematic. For the relatively simple SDEs arising here, the analysis is straightforward and may be computed to high-order. We deduce that these noises may be replaced by appropriately chosen drift and white noise terms.

In summary, in this paper we bring the above separate threads together. We show how to combine the best features of each to develop an efficient and practical approach for low-dimensional modelling of dynamical systems. The ideas are developed within the context of a simple modelling problem. We do this to pinpoint what is achieved and what needs to be done, without obscuring the issues with complicating algebraic generalities.

Throughout, we adopt the Stratonovich interpretation of SDEs.

2 Direct reduction

In this section we demonstrate that the standard form of the algorithm to find a forced centre manifold, as derived by Cox & Roberts [7], when extended to higher order in the forcing amplitude and applied to Stratonovich SDEs produces the same results as obtained by Schöner & Haken [20] using the slaving principle (the similarity in the basic approaches has previously been commented on, for example by Wunderlin & Haken [23]). However, as in the method of normal forms [12, §4.2] and unlike simpler methods, the centre manifold approach easily caters for nilpotent linearisations (for example, the smallness assumption on the linearisation of the master modes of the slaving principle, that $\Lambda_u \sim \delta$ in (3.2) of [20], is unnecessarily restrictive), and is readily generalised to construct accurate invariant manifolds [16]. Further, the great advantage the centre manifold approach has over the normal form procedure of Sri Namachchcivaya & Lin [12, §4.2] is that one need only compute the low-dimensional centre manifold; unlike normal form calculations, no algebra is done to describe the uninteresting details of the large number, possibly infinite in number [11, 19, e.g.], of exponentially decaying modes.

The dynamical model we develop in this section provides basic results for later comparison and improvement.

2.1 Formal elimination near a pitchfork bifurcation

Consider the following example SDE discussed by Schöner & Haken [20, §5]

$$dx = (\alpha x - axy)dt + F_x dW_1\,, \tag{1}$$

$$dy = (-\beta y + bx^2)dt + F_y dW_2\,, \tag{2}$$

where W_1 and W_2 are independent Wiener processes, and α, β, a and b are real constants (with a and b having the same sign). In the absence of the stochastic forcing this dynamical system undergoes a pitchfork bifurcation as the parameter α crosses 0; here we may be interested in investigating effects upon the bifurcation of the additive noise.

Near the bifurcation, for small α, $x(t)$ evolves slowly whereas $y(t)$ decays exponentially to zero. Thus x is the order parameter and y the slaved process. More rigorously, we append the dynamical equation $d\alpha = 0dt$ and claim that Theorem 6.1 of Boxler [4] asserts that there exists a local stochastic centre manifold for (1–2) of the form $y = h(t, x, \alpha)$ for x and α small enough.

Thus we assume the stochastic centre manifold (SCM) is

$$y = h(t, x)\,, \tag{3}$$

where the dependence upon α is implicit, and where we take the slaved variable to explicitly depend on only two variables: time; and the order parameter x. Schöner & Haken [20, Eq.(2.1)] instead used many variables, $y = h\left(t, x, Z_\nu\right)$, where $Z_\nu(t)$ denotes a large number of as yet unknown noise processes which appear in the low-dimensional model. Here we considerably simplify the algebra by including, albeit implicitly, the effects of such noise through the time variable t. Consequently, integrals with respect to time are performed treating x as a constant, but integrating any noise.

Differentiate (3) with respect to t to

$$dy = \frac{\partial h}{\partial t}dt + \frac{\partial h}{\partial x}dx\,. \tag{4}$$

Substituting (4), (2) is written in the following form:

$$\left[\frac{\partial}{\partial t}dt + \beta I dt\right] h \;=\; bx^2 dt + F_y dW_2 - \frac{\partial h}{\partial x}dx\,. \tag{5}$$

Substituting for dx from (1) we obtain:

$$\mathcal{B}h \;=\; bx^2 dt + F_y dW_2 - \frac{\partial h}{\partial x}\left[(\alpha x - axh)dt + F_x dW_1\right]\,, \tag{6}$$

where

$$\mathcal{B} = \frac{\partial}{\partial t}dt + \beta I\, dt\,.$$

This equation is to be solved for h, the SCM, albeit approximately.

In the following we solve a sequence of problems of the form

$$\mathcal{B}g = f_1 dt + f_2 dW\,, \tag{7}$$

where $f_1(t, x)$ and $f_2(t, x)$ are known functions, and we need to determine the process $g(t, x)$. We may deduce[20, Eq.(2.20)]

$$g = e^{-\beta t} \star \left(f_1 dt + f_2 dW\right) = \int_{-\infty}^{t} e^{-\beta(t-\tau)}\left(f_1 d\tau + f_2 dW(\tau)\right)\,,$$

where "$\star$" denotes the given convolution. Note these convolution integrals are done keeping x constant as that is implicit in the notation $\partial/\partial t$ of time differentiation, whereas all noise processes appearing in the integrand must be integrated.

2.2 Asymptotic solution

To simplify the solution procedure we define δ to be

$$\delta^2 = \|x(t)\|^2 + |F_x| + |F_y| + |\alpha| \,,$$

so that δ is a quantitative measure of the overall size of the solution and contributing terms. In effect, for small δ the following scaling relations hold:

$$x = \mathcal{O}\left(\delta\right) \,, \qquad \alpha,\ F_x,\ F_y \text{ are } \mathcal{O}\left(\delta^2\right) \,,$$

whereas the other constants are of order 0 in δ. In principle, the centre manifold formalism allows us to expand the solution in terms of the parameters independently. However, as discussed in [14], by using one ordering parameter δ we obtain the same set of terms, it is just that they appear in the asymptotic expansion in a regimented order. Thus we expand

$$h \;\sim\; \sum_{m=2}^{\infty} h^{(m)}(t,x) \,, \tag{8}$$

where $h^{(m)}$ contains all terms of order m in δ. Substituting into (6), we obtain

$$\begin{aligned} \mathcal{B}\sum_{m=2}^{\infty} h^{(m)}(t,x) \;&\sim\; bx^2dt + F_y dW_2 \\ &\quad - \sum_{m=2}^{\infty} \frac{\partial h}{\partial x}^{(m)} \left[\left(\alpha x - ax\sum_{m=2}^{\infty} h^{(m)}\right) dt + F_x dW_1\right] . \end{aligned} \tag{9}$$

- Extracting all terms of order $m = 2$,

$$\begin{aligned} h^{(2)}(t,x) &= \mathcal{B}^{-1}\left[bx^2dt + F_y dW_2\right] \\ &= e^{-\beta t} \star \left[bx^2dt + F_y dW_2\right] \\ &= \frac{b}{\beta}x^2 + F_y Z^{(2)} \,, \end{aligned} \tag{10}$$

where

$$Z^{(2)}(t) = \int_{-\infty}^{t} e^{-\beta(t-\tau)} dW_2(\tau) = e^{-\beta t} \star dW_2 \,. \tag{11}$$

- Terms of order $m = 3$ similarly lead to

$$h^{(3)}(t,x) = -\frac{2b}{\beta} x F_x Z^{(3)} , \tag{12}$$

where

$$Z^{(3)} = e^{-\beta t} \star dW_1 . \tag{13}$$

- Terms of order $m = 4$ give

$$h^{(4)}(t,x) = -\frac{2\alpha b x^2}{\beta^2} + \frac{2ab^2x^4}{\beta^3} + \frac{2b}{\beta} {F_x}^2 Z_1^{(4)} + \frac{2abx^2}{\beta} F_y Z_2^{(4)} , \tag{14}$$

where

$$Z_1^{(4)} = e^{-\beta t} \star Z^{(3)} dW_1 , \qquad \text{and} \qquad Z_2^{(4)} = e^{-\beta t} \star Z^{(2)} dt . \tag{15}$$

- Terms of order $m = 5$ give

$$\begin{aligned} h^{(5)}(t,x) &= \frac{2\alpha b}{\beta} x F_x Z_3^{(5)} - \frac{6ab^2}{\beta^2} x^3 F_x Z_3^{(5)} - \frac{2abx}{\beta} F_x F_y Z_4^{(5)} \\ &\quad + \left(\frac{4\alpha b x}{\beta^2} - \frac{8ab^2x^3}{\beta^3} \right) F_x Z_1^{(5)} - \frac{4abx}{\beta} F_x F_y Z_2^{(5)} , \end{aligned} \tag{16}$$

where

$$\begin{aligned} Z_1^{(5)} &= e^{-\beta t} \star dW_1 \, \left(= Z^{(3)} \right) , & Z_2^{(5)} &= e^{-\beta t} \star Z_2^{(4)} dW_1 , \\ Z_3^{(5)} &= e^{-\beta t} \star Z^{(3)} dt , & Z_4^{(5)} &= e^{-\beta t} \star Z^{(2)} Z^{(3)} dt . \end{aligned} \tag{17}$$

Clearly this procedure may be continued indefinitely to derive, in principle, arbitrarily high-order asymptotic approximations to the stochastically forced centre manifold.

After substituting the approximate $y = h(t,x)$ into the x equation, we get the following stochastic evolution equation, accurate to 6th order in δ:

$$\begin{aligned} dx \sim \; & F_x dW_1 + \left\{ \left[\alpha x - \frac{ab}{\beta} x^3 - a F_y Z^{(2)} x \right] + \left[\frac{2ab}{\beta} F_x Z^{(3)} x^2 \right] \right. \\ & + \left[\frac{2a\alpha b}{\beta^2} x^3 - \frac{2a^2b^2}{\beta^3} x^5 - \frac{2a^2b}{\beta} F_y Z_2^{(4)} x^3 - \frac{2ab}{\beta} {F_x}^2 Z_1^{(4)} x \right] \\ & + \left[-\frac{2a\alpha b}{\beta} F_x Z_3^{(5)} x^2 + \frac{6a^2b^2}{\beta^2} F_x Z_3^{(5)} x^4 + \frac{2a^2b}{\beta} F_x F_y Z_4^{(5)} x^2 \right. \\ & \left. \left. - \left(\frac{4a\alpha b}{\beta^2} x^2 - \frac{8a^2b^2}{\beta^3} x^4 \right) F_x Z_1^{(5)} + \frac{4a^2b}{\beta} F_x F_y Z_2^{(5)} x^2 \right] \right\} dt . \end{aligned} \tag{18}$$

This last SDE describes the low-dimensional evolution relevant to the noisy pitchfork bifurcation. As mentioned earlier, these results are identical to those of Schöner & Haken [20, Eq.(5.15)], but the derivation is shorn of unnecessary complicating detail and in the framework of the more powerful centre manifold approach.

3 Simplify the noise processes

One of the remarkable features of the preceding analysis is the appearance of many noise processes in the low-dimensional model (18). In this example there are 8 new noises up to fifth order; in interesting physical examples there would be many more corresponding to each of the neglected transient modes (see (28–29) in [6] for example). Further, each of these noise processes involves a memory, they are each a convolution integral over the past history of a noise source. In the context of simply forced dynamical systems, Cox & Roberts [7] showed how to adapt the centre manifold procedure to eliminate the memory effect and consequently simplify the forcing in low-dimensional models; however, the analysis was only linear in the amplitude of the forcing. In a similar spirit, Coullet *et al* [6] and later Sri Namachchivaya & Lin [13] examined additive and linearly-multiplicative noise, using the normal form transformation to limit the complexity of not only the deterministic terms, but also to simplify the noise on the centre and the stable manifolds.

In this section we show that the approach of Cox & Roberts [7], in conjunction with new ideas on the removal of resonant stochastic terms, leads directly to a simple model on the centre manifold; thus simplifying the methods and results compared with those obtainable with the methods of previous authors. The two critical steps are to allow some flexibility in the parameterization of the centre manifold and to examine the stochastic resonances directly rather than in Fourier space. In doing so, as discussed in [7], we implicitly cater for the nature of the evolution near to the centre manifold and consequently discover how to treat the stochastic forcing in a simple manner. Again we introduce the ideas as applied to the simple system (1–2).

3.1 Reparameterise the centre manifold

Assume the centre manifold is described by

$$x = s + \xi(t, s)\,, \tag{19}$$

$$
\begin{aligned}
y &= \eta(t,s)\,, && (20)\\
ds &= g(t,s) = g_0(t,s)dt + g_1(t,s)dW_1 + g_2(t,s)dW_2\,, && (21)
\end{aligned}
$$

where s parameterizes the centre manifold and hence $s(t)$ describes the evolution of the low-dimensional model. Note that (19) is a near identity transformation of x and we use the flexibility in ξ to eliminate as far as possible unnecessary noise. The analysis is in the spirit of normal forms; however, here we directly construct a normal form on the centre manifold without going through the laborious task of finding a normal form of the entire system. Substitute (20) and (19) into the y and x evolution equations, (2) and (1), to get:

$$
\begin{aligned}
dy &= \frac{\partial\eta}{\partial t}dt + \frac{\partial\eta}{\partial s}ds = \left[-\beta\eta + b(s+\xi)^2\right]dt + F_y dW_2\,, && (22)\\
dx &= \frac{\partial\xi}{\partial t}dt + \left(1+\frac{\partial\xi}{\partial s}\right)ds = \left[\alpha s + \alpha\xi - a(s+\xi)\eta\right]dt + F_x dW_1\,, && (23)
\end{aligned}
$$

and hence, using (21),

$$
\begin{aligned}
\mathcal{B}\eta &= b(s^2+2s\xi+\xi^2)dt + F_y dW_2 - \frac{\partial\eta}{\partial s}g\,, && (24)\\
g &= -\frac{\partial\xi}{\partial t}dt - \frac{\partial\xi}{\partial s}g + \left[\alpha s + \alpha\xi - as\eta - a\xi\eta\right]dt + F_x dW_1\,. && (25)
\end{aligned}
$$

Substitute the asymptotic expansions

$$
\begin{aligned}
\eta &\sim \eta^{(2)} + \eta^{(3)} + \eta^{(4)} + \cdots\,,\\
\xi &\sim \xi^{(2)} + \xi^{(3)} + \xi^{(4)} + \cdots\,,\\
g &\sim g^{(2)} + g^{(3)} + g^{(4)} + \cdots\,,
\end{aligned}
$$

where, as before, the superscript $^{(n)}$ denotes quantities of order n in δ (note that $s = \mathcal{O}(\delta)$). After comparison of the order of the terms appearing in (24) and (25), we obtain the following equations and their solution.

3.2 Order 2: simple resonance

Terms of order $m = 2$ give:

$$
\mathcal{B}\eta^{(2)} = bs^2dt + F_y dW_2\,, \quad \text{and} \quad g^{(2)} = -\frac{\partial\xi}{\partial t}^{(2)}dt + F_x dW_1\,. \qquad (26)
$$

It is not reasonable to use the freedom in $\xi^{(2)}$ to eliminate dW_1 in $g^{(2)}$ in order to simplify the model SDE $ds = g$, therefore

$$\xi^{(2)} = 0\,, \qquad \eta^{(2)} = \frac{b}{\beta} s^2 + F_y Z^{(2)}\,, \quad \text{and} \quad g^{(2)} = F_x dW_1\,, \tag{27}$$

where $Z^{(2)} = e^{-\beta t} \star dW_2$ as in (11). If one attempts to use $\xi^{(2)}$ to eliminate the noise $F_x dW_1$, then one is lead to $\xi^{(2)} = W_1$ which over long-times grows like $t^{1/2}$. Such secular growth, albeit stochastic, leads to an unallowable non-uniform convergence in time: eventually $\xi^{(2)}$ would no longer be a small perturbation to the leading-order term in $x = s+\xi(t,s)$. Thus the component in dW_1 must remain in g. This sort of direct argument about long-time behaviour, rather than resonance in Fourier space, determines which terms we may eliminate through the reparameterization (normal form transformation) of the centre manifold.

3.3 Order 3: coloured noise

Terms of order $m = 3$ lead to:

$$\mathcal{B}\eta^{(3)} = -\frac{2b}{\beta} s F_x dW_1\,, \qquad \text{and} \qquad g^{(3)} = \left(\alpha s - \frac{ab}{\beta} s^3 - \frac{\partial \xi}{\partial t}^{(3)} - a s F_y Z^{(2)}\right) dt\,.$$

$\xi^{(3)}$ cannot be used to eliminate either the αs nor the s^3 terms as they have non-zero mean (over t, keeping s constant), and hence would generate secular growth if they were included in $\xi^{(3)}$. Also, we would like $\frac{\partial \xi}{\partial t}^{(3)} + a s F_y Z^{(2)} = 0$, namely $\xi^{(3)} = -a s F_y \int Z^{(2)} dt$, in order to simplify the above equation for $g^{(3)}$; however, this is not possible as we now explain. From (11), the coloured noise $Z^{(2)}$ satisfies the SDE

$$dZ^{(2)} = -\beta Z^{(2)} dt + dW_2\,. \tag{28}$$

Thus over long-times, on the time-scales we are interested in for the low-dimensional model, namely those that are much longer than $1/\beta$, $Z^{(2)} = \mathcal{O}\left(t^{-1/2}\right)$ as $dt = \mathcal{O}(t)$ and $dW_2 = \mathcal{O}\left(t^{1/2}\right)$. Consequently, $\int Z^{(2)} dt = \mathcal{O}\left(t^{1/2}\right)$ is unallowably secular, and hence it would appear, as in [6, 13], that the $Z^{(2)} dt$ noise must remain in the model SDE $ds = g$.

Nonetheless, as in the analysis of deterministic forced systems by Cox & Roberts [7], we can usefully simplify the model SDE by removing the "colour"

of the noise, that is we avoid the memory intrinsic to $Z^{(2)}$ and processes like it. But first a digression.

One of the benefits of only considering the dynamics on the centre manifold, apart from its low-dimensionality, is that often the evolution is intrinsically slow; and even when the evolution is characterised by fast oscillation, as in a Hopf bifurcation [6] for example, a normal form transformation renders the evolution in terms of slowly-varying amplitudes. Thus we expect to only focus our attention on the long-time scales of the model—for example, one may take long time-steps in a numerical solution—and we do not need to resolve any details of the rapidly decaying transients. But the form of the evolution (18) on the stochastic centre manifold involves noise processes $Z^{(\nu)}(t)$ which are coloured on the fast-time scale of the decaying transients. This can be seen clearly in (28), and similarly for the other noises introduced in the previous section. Here each noise process is dominated by the decay on the fast-time scale of $1/\beta$. Precisely the same effect can be seen in the normal form results of Sri Namachchivaya [13], Eqs.(14–15) in general and Eq.(39) in particular, where the noise on the centre manifold modes, U_{cs} in general, decays according to the linearised operator of the decaying modes. It is also seen in Eq.(29) of Coullet *et al* [6]. However, the latter also point out that on the long-time scales of the model, a coloured noise may be approximated by a white noise, here $Z^{(2)}dt \approx dW_2/\beta$: a result easily identified from a balance of terms in (28). We now proceed to show how to turn this simplifying approximation into an exact result.

Rewrite the SDE (28) for $Z^{(2)}$ in the form

$$Z^{(2)}dt = -\frac{dZ^{(2)}}{\beta} + \frac{dW_2}{\beta}\,,$$

whence it follows that we can replace $Z^{(2)}dt$ by the above in the right-hand side of the equation for $g^{(3)}$ to obtain

$$g^{(3)} = \left(\alpha s - \frac{ab}{\beta}s^3\right)dt - \frac{\partial\xi}{\partial t}^{(3)}dt + \frac{a}{\beta}sF_y dZ^{(2)} - \frac{a}{\beta}sF_y dW_2\,.$$

Now the term in $dZ^{(2)}$ can be absorbed into $\xi^{(3)}$, leaving just the white noise dW_2 in $g^{(3)}$ for the model SDE. Thus

$$\begin{aligned} &\xi^{(3)} = \tfrac{a}{\beta}sF_yZ^{(2)}\,, \quad \eta^{(3)} = -\tfrac{2b}{\beta}sF_xZ^{(3)}\,, \\ \text{and}\qquad &g^{(3)} = \left(\alpha s - \tfrac{ab}{\beta}s^3\right)dt - \tfrac{a}{\beta}sF_ydW_2\,, \end{aligned} \tag{29}$$

where $Z^{(3)} = e^{-\beta t} \star dW_1$ as before in (13).

In summary, to third order we deduce the centre manifold is

$$\begin{aligned} x &\sim s + \frac{a}{\beta} s F_y Z^{(2)} \,, \\ y &\sim \frac{b}{\beta} s^2 + F_y Z^{(2)} - \frac{2b}{\beta} s F_x Z^{(3)} \,, \\ \text{s.t.} \quad ds &\sim \left(\alpha s - \frac{ab}{\beta} s^3 \right) dt + F_x dW_1 - \frac{a}{\beta} s F_y dW_2 \,. \end{aligned}$$

This is an exact decomposition: to this order we have put all the "colour" into the description of the location and shape of the stochastically forced centre manifold, leaving just pure white noise in the model evolution equation.

3.4 Order 4: nonlinearly generated noise

Equating terms of order $m = 4$ in (24–25) give:

$$\begin{aligned} \mathcal{B}\eta^{(4)} &= \frac{2ab}{\beta} s^2 F_y Z^{(2)} dt - \frac{2b}{\beta} s \left(\alpha s - \frac{ab}{\beta} s^3 \right) dt + \frac{2b}{\beta} F_x^2 Z^{(3)} dW_1 \\ &\quad + \frac{2ab}{\beta^2} s^2 F_y dW_2 \,, \\ g^{(4)} &= -\frac{\partial \xi^{(4)}}{\partial t} dt - \frac{a}{\beta} F_x F_y Z^{(2)} dW_1 + \frac{2ab}{\beta^2} s^2 F_x dW_1 \,. \end{aligned}$$

The terms to the right of $\xi^{(4)}$ are resonant, $dW_1 = \mathcal{O}\left(t^{1/2}\right)$ and $Z^{(2)} dW_1 = \mathcal{O}\left(t^0\right)$, so that at least some component of these noises has to be assigned to $g^{(4)}$. At this stage we cannot think of a useful transformation of the $Z^{(2)} dW_1$ contribution, and so must choose

$$\begin{aligned} \xi^{(4)} &= 0 \,, \quad \text{and thus} \\ \eta^{(4)} &= -\frac{2b}{\beta^2} s \left(\alpha s - \frac{ab}{\beta} s^3 \right) + \frac{2ab}{\beta} F_y s^2 \left(Z_2^{(4)} + \frac{Z^{(2)}}{\beta} \right) + \frac{2b}{\beta} F_x^2 Z_1^{(4)} \,, \qquad (30) \\ g^{(4)} &= +\frac{2ab}{\beta^2} s^2 F_x dW_1 - \frac{a}{\beta} F_x F_y Z^{(2)} dW_1 \,, \end{aligned}$$

where $Z_1^{(4)} = e^{-\beta t} \star Z^{(3)} dW_1$ and $Z_2^{(4)} = e^{-\beta t} \star Z^{(2)} dt$ as before, (15).

A prime target for further research is to simplify the nonlinear combination of $Z^{(2)}dW_1$; the difficulty is that this is not a normal coloured noise term. Its presence here shows that the typical assumption of Gaussian noise is generally not justified in modelling nonlinear dynamics. However, in the next section we outline a Fokker-Planck approach to the rational simplification of such nonlinear noise processes.

3.5 Order 5: noise induced drift

Terms of order $m = 5$ lead to:

$$\begin{aligned}
\mathcal{B}\eta^{(5)} &= \frac{2b}{\beta}(\alpha s - \frac{3ab}{\beta}s^3)F_x Z^{(3)}dt - \frac{2ab}{\beta^2}sF_xF_yZ^{(3)}dW_2 \\
&\quad + \left[\frac{4b}{\beta^2}s(\alpha - \frac{3ab}{\beta}s^2) - \frac{4ab}{\beta}sF_yZ_2^{(4)} - \frac{2ab}{\beta^2}sF_yZ^{(2)}\right]F_x dW_1\,, \\
g^{(5)} &= -\frac{\partial\xi^{(5)}}{\partial t}dt + \frac{2\alpha ab}{\beta^2}s^3dt - \frac{2a^2b^2}{\beta^3}s^5dt - \frac{2a^2b}{\beta}s^3F_y\left(Z_2^{(4)} + \frac{1}{\beta}Z^{(2)}\right)dt \\
&\quad - \frac{2ab}{\beta}sF_x^2Z_1^{(4)}dt - \frac{a^2}{\beta}sF_y^2{Z^{(2)}}^2dt + \frac{4ab}{\beta^2}sF_x^2Z^{(3)}dW_1 + \frac{a^2}{\beta^2}sF_y^2Z^{(2)}dW_2\,.
\end{aligned}$$

As before we use $\xi^{(5)}$ to absorb as much of the evolution terms in $g^{(5)}$. Recall that $Z^{(2)}dt = -\frac{1}{\beta}dZ^{(2)} + \frac{1}{\beta}dW_2$, so that

$${Z^{(2)}}^2dt = Z^{(2)}\left(Z^{(2)}dt\right) = -\frac{1}{2\beta}d\left({Z^{(2)}}^2\right) + \frac{1}{\beta}Z^{(2)}dW_2\,,$$

whereas from (15)

$$\begin{aligned}
Z_2^{(4)}dt &= -\frac{1}{\beta}dZ_2^{(4)} + \frac{1}{\beta}Z^{(2)}dt = -\frac{1}{\beta}dZ_2^{(4)} - \frac{1}{\beta^2}dZ^{(2)} + \frac{1}{\beta^2}dW_2\,, \\
Z_1^{(4)}dt &= -\frac{1}{\beta}dZ_1^{(4)} + \frac{1}{\beta}Z^{(3)}dW_1\,.
\end{aligned}$$

Thus we set

$$\begin{aligned}
\xi^{(5)} &= +\frac{2a^2b}{\beta^2}s^3F_y\left(Z_2^{(4)} + \frac{2}{\beta}Z^{(2)}\right) + \frac{2ab}{\beta^2}sF_x^2Z_1^{(4)} + \frac{a^2}{2\beta^2}sF_y^2{Z^{(2)}}^2\,, \\
\eta^{(5)} &= \frac{2b}{\beta}\left(\alpha s - \frac{3ab}{\beta}s^3\right)F_x\left(Z_3^{(5)} + \frac{2}{\beta}Z^{(3)}\right) - \frac{4ab}{\beta}sF_xF_yZ_2^{(5)} - \frac{2ab}{\beta^2}sF_xF_yZ_5^{(5)}\,, \\
g^{(5)} &= \left(\frac{2\alpha ab}{\beta^2}s^3 - \frac{2a^2b^2}{\beta^3}s^5\right)dt - \frac{4a^2b}{\beta^3}s^3F_ydW_2 + \frac{2ab}{\beta^2}sF_x^2Z^{(3)}dW_1\,, \qquad (31)
\end{aligned}$$

where the new noise

$$dZ_5^{(5)} = e^{-\beta t} \star \left(Z^{(2)} dW_1 + Z^{(3)} dW_2 \right) . \tag{32}$$

A factor of interest in $g^{(5)}$ is the non-Gaussian noise $Z^{(3)}dW_1$ which has non-zero mean and predicts a long-term "drift" in the low-dimensional model of the order of the square of the stochastic forcing. This term, being linearly multiplicative in s will contribute to a destabilisation of the origin.

3.6 Asymptotic model

Therefore, we obtain ds to fifth order as follows:

$$\begin{aligned} ds \sim & \left(\alpha s - \frac{ab}{\beta} s^3 + \frac{2\alpha ab}{\beta^2} s^3 - \frac{2a^2b^2}{\beta^3} s^5 \right) dt \\ & + \left(1 + \frac{2ab}{\beta^2} s^2 \right) F_x dW_1 - \left(\frac{a}{\beta} s + \frac{4a^2 b}{\beta^3} s^3 \right) F_y dW_2 \\ & - \frac{a}{\beta} F_x F_y Z^{(2)} dW_1 + \frac{2ab}{\beta^2} s F_x^2 Z^{(3)} dW_1 . \end{aligned} \tag{33}$$

This has significantly simpler structure than the fifth order version of Schöner & Haken that we derived in the previous section, namely (18) truncated to

$$\begin{aligned} dx \sim & F_x dW_1 + \left\{ \left[\alpha x - \frac{ab}{\beta} x^3 - a F_y Z^{(2)} x \right] + \left[\frac{2ab}{\beta} F_x Z^{(3)} x^2 \right] \right. \\ & \left. + \left[\frac{2a\alpha b}{\beta^2} x^3 - \frac{2a^2b^2}{\beta^3} x^5 - \frac{2a^2 b}{\beta} F_y Z_2^{(4)} x^3 - \frac{2ab}{\beta} {F_x}^2 Z_1^{(4)} x \right] \right\} dt . \end{aligned}$$

There are only two new noise processes in the model, instead of four, and the two noises which are new are obtained from lower order SDEs. Instead we see in (33) a richer structure directly in terms of the noise from the original system.

4 Fokker-Planck analysis of nonlinear noise

In this section we turn to the appearance of the noise processes $Z^{(3)}dW_1$ and $Z^{(2)}dW_1$ in the low-dimensional model (33). The computation of $Z^{(2)}$ and $Z^{(3)}$, via the convolutions in (11) and (13), necessarily takes place on the fast

time-scale $1/\beta$, and must therefore be a major hindrance to the use of the model over long times. Such fast time-scales in the noise are not apparent in the centre manifold analysis of Fokker-Planck equations by Knobloch & Wiesenfeld [10]. In a pitchfork bifurcation, for example, the evolution on the centre manifold of a Fokker-Planck equation [10, Eq.(3.22)] is equivalent to the stochastic differential equation

$$du = \left(\mu u + a u^3\right) dt + \sigma d\xi\,,$$

where $d\xi(t)$ represents Gaussian white noise. In simpler systems one can carry out an analysis of the Fokker-Planck equations to high order and see that while the effective noise may be non-Gaussian, there is no remnant of the fast time-scale. We show this here, and then suggest how to model $Z^{(3)}dW_1$ and $Z^{(2)}dW_1$.

We propose to replace these nonlinearly generated noises with new independent Wiener processes with the same long-term statistics. In essence we construct a *weak* low-dimensional model by these arguments—fidelity with the original dynamics can only be assured via the undesired evaluation of fast-time integrals.

4.1 Self excited drift

In the SDE (33), we would like to replace the factor $Z^{(3)}dW_1$ by a dw where dw represents some simple noise process which could be sampled over relatively large times. That is, using the definition of $Z^{(3)}$, we seek to understand something of the evolution of $w(t)$ in

$$dw = Z^{(3)}dW_1\,, \quad dZ^{(3)} = -\beta Z^{(3)}dt + dW_1\,.$$

This is in the form of a SDE with one neutral mode w and one exponentially decaying mode $Z^{(3)}$. Thus we may apply centre manifold techniques to derive a model SDE for the mode $w(t)$. However, if this is done using the techniques described earlier, then one finds exactly the same type of noise process in the results. The above pair of coupled SDEs is in some sense canonical in that it is irreducible using the previous techniques. Instead we turn to the Fokker-Planck equation; we must do something different because the results, equation (41), involve the surd $\sqrt{\beta}$ which is impossible to obtain via the direct application of the algebra of centre manifold techniques.

For convenience we rewrite the SDE as

$$dx = \epsilon y dW_1 \,, \quad dy = -\beta y dt + \epsilon dW_1 \,, \tag{34}$$

where, in this subsection *only*, $x = w$ and $y = Z^{(3)}$. The "small" parameter ϵ has been introduced to order the terms in the asymptotic analysis; eventually we set $\epsilon = 1$ to recover approximate results for the original process. The Fokker-Planck equation of this Stratonovich SDE is

$$\partial_t u = \frac{\epsilon^2}{2}\partial_x u + \beta \partial_y (yu) + \frac{\epsilon^2}{2}\left\{ y^2 \partial_{xx} u + 2\partial_{xy}(yu) + \partial_{yy} u \right\} \,, \tag{35}$$

for the probability density function $u(x, y, t)$. We seek a model for the long-term evolution of $x(t)$ via the derivation of a Fokker-Planck equation for the probability density function

$$p(x,t) = \int_{-\infty}^{\infty} u(x,y,t)\, dy \,. \tag{36}$$

Heuristically, the term $\beta\partial_y (yu)$ in (35) concentrates u upon $y = 0$, but this is countered by the noise induced diffusion term $\frac{\epsilon^2}{2}\partial_{yy} u$. Thus to leading order, u is Gaussian in y—but very localised if ϵ is small. Then the other terms in the Fokker-Planck equation cause a long-term drift and spread in the x direction which we describe in terms of $p(x, t)$.

We use centre manifold techniques to find the long-term evolution of p. Because of the symmetry and simplicity of (34) we can analyse the Fokker-Planck equation somewhat more straightforwardly than Knobloch & Wiesenfeld [10]. First, scale

$$y = \epsilon Y \,, \quad u = \frac{1}{\epsilon} U(x, Y, t) \,,$$

so that with Y we can resolve the Gaussian structure, and any modifications, in y. Then seek

$$U = V(Y,p) \sim \sum_{n=0}^{\infty} \epsilon^n V^{(n)}(Y,p) \,, \tag{37}$$

$$\text{such that} \quad \partial_t p = g(p) \sim \sum_{n=0}^{\infty} \epsilon^n g^{(n)}(p) \,. \tag{38}$$

Substituting these into (35) and (36) gives a hierarchy of equations to solve for $V^{(n)}$ and $g^{(n)}$. Using computer algebra it is straight-forward to find

$$\begin{aligned} U &\sim G(Y)p \\ &\quad +\epsilon^2\left(-Y^2+\frac{1}{2\beta}\right)G(Y)p_x \\ &\quad +\epsilon^4\left(\frac{1}{2}Y^4-\frac{1}{4\beta}Y^2-\frac{1}{4\beta^2}\right)G(Y)p_{xx} \\ &\quad +\epsilon^6\left(-\frac{1}{6}Y^6+\frac{1}{8\beta^2}Y^2+\frac{1}{4\beta^3}\right)G(Y)p_{xxx}\,, \end{aligned} \tag{39}$$

$$\text{such that} \quad p_t \sim -\frac{\epsilon^2}{2}p_x+\frac{\epsilon^4}{4\beta}p_{xx}-\frac{\epsilon^6}{4\beta^2}p_{xxx}\,, \tag{40}$$

where $G(Y)=\sqrt{\beta/\pi}\exp\left(-\beta Y^2\right)$ is the leading-order Gaussian structure.

This last evolution equation (40), when truncated to neglect terms $o\left(\epsilon^4\right)$, is the Fokker-Planck equation of a SDE such as

$$dx \sim \frac{\epsilon^2}{2}dt+\frac{\epsilon^2}{2\sqrt{\beta}}dW\,, \tag{41}$$

where the noise process dW may be considered essentially independent of the original noise dW_1 because, on the long-time scales that we wish to use these noises, the nonlinear process generating $x(t)$ summarises quite different information about the details of dW_1 than that of the simple cumulative sum seen in $W_1(t)$. This is confirmed by numerical simulations. Figure 1 shows that, on time scales long compared with the decay time $1/\beta$, there is virtually no linear correlation between the noises dW_1 and $dx-dt/2$; there are nonlinear correlations for smaller times, e.g. $\Delta t=2/\beta$, but these slowly disappear, as seen for $\Delta t=16/\beta$. Thus we propose to replace, in the low-dimensional model (33), the fast time-scale factors $Z^{(3)}dW_1$ by the above dx for $\epsilon=1$, namely by $\frac{1}{2}dt+\frac{1}{2}dW_3/\sqrt{\beta}$ where W_3 is a new independent Wiener process. In essence, this replacement summarises the drift and overall fluctuations in $Z^{(3)}dW_1$ while ignoring the other details of the nonlinear noise.

There are two issues to comment on. Firstly, it is reasonable to truncate (40) to the first two terms because this is the lowest order structurally stable approximation to the evolution of $p(x,t)$. Further, although there are higher-order modifications, it is apparent that $g^{(2n)}=\mathcal{O}\left(\beta^{1-n}\right)$ so that, provided the decay of the "slaved" modes are rapid enough, these higher-order terms may be neglected while maintaining accuracy.

Figure 1: scatter plots of $\Delta(x-t/2)$ versus ΔW_1 on two time-scales: left-hand plot for $\Delta t = 2/\beta$; right-hand plot for $\Delta t = 16/\beta$. The data is generated by public domain software, Gnans, numerically integrating (34) over $0 \leq t \leq 10,000$ with a time-step of 0.01.

Secondly, strictly speaking the asymptotic expansion (39) is not uniformly valid and so could be in error; it is evident that $V^{(n)} \propto Y^n G(Y)$ and so $V^{(n)} \gg V^{(n-1)}$ for large enough $|Y|$. However, instead of solving for all Y we may restrict analysis to $|Y| < \Psi = \mathcal{O}\left(\epsilon^{-1/2}\right)$, that is $|y| = \mathcal{O}\left(\sqrt{\epsilon}\right)$, then

- the errors in doing this are typically $\mathcal{O}\left(G(\Psi)\right) = \mathcal{O}\left(\exp(-\beta/\epsilon)\right)$, namely exponentially small;
- and for large $Y < \Psi$, at worst $V^{(n+1)} = \mathcal{O}\left(\sqrt{\epsilon} V^{(n)}\right)$ which preserves the ordering of the asymptotic terms.

Consequently, the above expansion is asymptotically valid.

4.2 Independent jittering

Similarly, in the SDE (33), we would like to replace the factor $Z^{(2)} dW_1$ by a dw where

$$dw = Z^{(2)} dW_1\,, \quad dZ^{(2)} = -\beta Z^{(2)} dt + dW_2\,.$$

Here the two noise processes, W_1 and W_2, are independent, thus causing the Fokker-Planck equation of this Stratonovich SDE to be the somewhat simpler

$$\partial_t u = \beta \partial_y (yu) + \frac{\epsilon^2}{2} \left\{ y^2 \partial_{xx} u + \partial_{yy} u \right\}\,, \tag{42}$$

Figure 2: scatter plots on the time-scale $\Delta t = 2/\beta$: left-hand plot of Δx versus ΔW_1; right-hand plot for Δx versus ΔW_2. As before, the data is generated by numerical integration over $0 \leq t \leq 10,000$ with a time-step of 0.01.

where here $x = w$ and $y = Z^{(2)}$. As before, we seek a model for the long-term evolution of $x(t)$ by substituting (37–38) into this equation to find

$$\begin{aligned} U &\sim G(Y)p \\ &\quad +\epsilon^4 \left(\frac{1}{4\beta} Y^2 - \frac{1}{8\beta^2} \right) G(Y) p_{xx} \qquad (43) \\ &\quad +\epsilon^8 \left(\frac{1}{32\beta^2} Y^4 + \frac{1}{32\beta^3} Y^2 - \frac{5}{128\beta^4} \right) G(Y) p_{xxxx} \,, \\ \text{such that} \quad p_t &\sim \frac{\epsilon^4}{4\beta} p_{xx} + \frac{\epsilon^8}{16\beta^3} p_{xxxx} \,. \qquad (44) \end{aligned}$$

This last evolution equation, when truncated to neglect terms $o(\epsilon^4)$, is the Fokker-Planck equation of a SDE such as

$$dx \sim \frac{\epsilon^2}{2\sqrt{\beta}} dW_4 \,, \qquad (45)$$

where the noise process dW_4 may be considered essentially independent of the other noises dW_1, dW_2 and dW_3. Again numerical experiments confirm this as seen by the lack of correlations in Figure 2. Here, this is so even on the relatively short time-scale of $\Delta t = 2/\beta$. There is no drift term in this SDE; the analysis suggests that over long time-scales we may treat the nonlinear noise $Z^{(2)} dW_1$ by a white noise $\frac{1}{2} dW_4/\sqrt{\beta}$.

4.3 Simplest, accurate, long-term model

The preceding analysis suggests that we may write the low-dimensional model

$$\begin{aligned} ds \sim & \left[\left(\alpha + \frac{ab}{\beta^2}F_x^2\right)s - \frac{ab}{\beta}s^3 + \frac{2\alpha ab}{\beta^2}s^3 - \frac{2a^2b^2}{\beta^3}s^5\right]dt \\ & + \left(1 + \frac{2ab}{\beta^2}s^2\right)F_x dW_1 - \left(\frac{a}{\beta}s + \frac{4a^2b}{\beta^3}s^3\right)F_y dW_2 \\ & - \frac{a}{2\beta^{3/2}}F_xF_y dW_4 + \frac{ab}{\beta^{5/2}}sF_x^2 dW_3 \,. \end{aligned} \tag{46}$$

This is the "simplest" model because:

- firstly, it is low-dimensional (here just one-dimensional instead of the original two-dimensions);
- secondly, it has no fast time-scale processes in it at all—other than those we call purely white noises whose long-term behaviour is known—and so may be numerically integrated using standard methods [9] with large time steps, ones appropriate to the phenomena of interest rather than ones dictated by the rapid time-scale of negligible modes.

Note that this model is a "weak" model unlike those of the previous sections. Previously, long-term fidelity of the model when compared with the original trajectories is assured by the algebra of the theory. However, here we have resorted, through the Fokker-Planck analysis, to using only broad statistics of $Z^{(3)}dW_1$ and $Z^{(2)}dW_1$ in the resultant model. The short-term details of $Z^{(3)}dW_1$ and $Z^{(2)}dW_1$, which are apparently needed to maintain fidelity, have been discarded. However, the processes W_1 and W_2 are invoked as noise precisely because we do not know their detailed dynamics, and so there is no loss in subsequently admitting that we do not know details of $Z^{(3)}dW_1$ and $Z^{(2)}dW_1$.

Throughout this analysis we have assumed that the original system was perturbed by white noise. However, the generalisation to "coloured" noise, Ornstein-Uhlenbeck processes, is straightforward, one just adjoins a dynamical equation describing the evolution of the coloured noise in terms of a new white noise. Consequently, we have shown how dynamical systems perturbed by even coloured noise may be systematically modelled by a low-dimensional system involving purely white noises.

On a philosophical level, we see in the process of reduction from (1–2) to (46) how unknown effects, represented by W_1 and W_2, effectively give rise to multiple noises in a mathematical model such as (46). Moreover, through the nonlinearities, the originally additive unknown processes naturally generate multiplicative noises.

5 Conclusion

This report has concentrated on a specific toy SDE. This is to demonstrate how crucial ideas can be developed and threaded together to form a low-dimensional modelling paradigm for SDEs in general. By avoiding complicated generalities we have been able to extend the centre manifold analysis to this interesting class of problems. Currently we are using the ideas described herein to investigate physical problems of significant interest. For example, we aim to investigate how stochastic effects, such as those due to turbulence, affect the dispersion of tracer in channels and pipes.

The analysis performed here has used the Stratonovich calculus for SDEs. As demonstrated by Schöner & Haken [21] the Ito analysis should produce completely equivalent results.

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